

PAR-LPTHE 02-39
hep-th/0207184

Going down from a 3-form in 16 dimensions

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Abstract: Group theory indicates the existence of a $SO(8) \times SO(7) \subset SO(16)$ invariant self-duality equation for a 3-form in 16 dimensions. It is a signal for interesting topological field theories, especially on 8-dimensional manifolds with holonomy group smaller than or equal to $Spin(7)$, with a gauge symmetry that is $SO(8)$ or $SO(7)$. Dimensional reduction also provides new supersymmetric theories in 4 and lower dimensions, as well as a model with gravitational interactions in 8 dimensions, which relies on the octonionic gravitational self-duality equation.

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1 Introduction

Self-duality equations play an important role in the context of topological field theory (TQFT), by providing topological gauge functions that one enforces in a BRST invariant way. This often determines supersymmetric actions in a twisted form. There is a classification in [1] of possible self-duality equations for the curvatures of forms of degree p in spaces with dimension d , $*G_{p+1} = T \wedge G_{p+1}$. Here, T is a tensor invariant under a maximal sub-group of $SO(d)$ and G_{p+1} is the curvature of the p -form. A requirement, for determining a TQFT for a form of a given degree, is that the number of self-duality equations for the curvature must equal the number of degrees of freedom of the form, modulo its gauge invariance (that is, the number of possible gauge covariant equations of motion for the form). This gives, case by case, and depending on the value of the space dimension, certain conditions for T , which were solved in [1], by using the available numerical Lie algebra tables. This numerical approach has limits the analysis to spaces with dimensions lower than 16, for forms of degrees less than 8. A certain number of non-trivial possibilities were found. They are listed in the table page 11 of reference [1]. This classification shows the existence of cases that go beyond the obvious self-duality equation $G_n = * G_n$ of forms of degree $n - 1$ in dimension $2n$, for which T is the $SO(2n)$ invariant antisymmetric tensor.

This table determines for instance the octonionic self-duality equation [2] for a Yang–Mills field in eight dimensions. The latter allows one to build the 8-dimensional Yang–Mills TQFT [3][4], which is $SO(8)$ covariant and $Spin(7)$ invariant. It is a twisted version of the 8-dimensional supersymmetric Yang–Mills theory and it exhibits a rich structure. By dimensional reduction and a suitable gauge-fixing in the Cartan algebra that is allowed by topological invariance, it gives the abelian monopole theory of Seiberg and Witten [5]. Other links with $N = 4, D = 4$ models have also been exhibited in [3], as well as links to matrix models.

The aim of this paper is to study the implications of another prediction of [1], namely, the existence of a self-duality equation for the curvature G_4 of a 3-form B_3 in 16 dimensions. This equation is invariant under a maximal subalgebra $SO(8) \times SO(7)$ of $SO(16)$. We will be interested in constructing 8-dimensional TQFT's that possibly descend from this self-duality equation, by dimensional reduction from 16 to 8 dimensions. They depend on fields coupled to the genuine 8-dimensional Yang–Mills theory [3], but, now, the

gauge symmetry group is determined. These fields descend from the 3-form, and their degrees of freedom can be spanned in suitable representations of $SO(8)$, or, possibly, of subgroups of $SO(8)$. We will directly build the curvatures in 8 dimensions. In TQFT's, the relationship between curvatures and forms is only restricted by the necessity of Bianchi identities. As a consequence of this freedom, one can get interactions in lower dimensions, although one starts from a free abelian 3-form in 16 dimensions. The determination of possible curvatures reduces to a rather easy algebraic problem, as indicated in earlier papers. Our procedure singles out two possible gauge symmetries, either $SO(8)$ or $SO(7)$ ¹. These gauge groups are quite relevant for the determination of instanton solutions. Indeed, in 8 dimensions, these groups play an analogous role to that of $SU(2)$ in 4 dimensions. The t'Hooft symbols $\eta_{\mu\nu}^a$, which mix the space and internal symmetry indices, and express the instanton solutions in four dimensions, are replaced in eight dimensions by other symbols, which are related to the octonion structure coefficients, as shown in [6]. G_2 is also an interesting possibility for relabeling the internal indices. By a further dimensional reduction from 8 to 4 dimensions, new couplings to matter can be found. By going down to two dimensions we suggest a connection with a Matrix theory description of the Seiberg–Witten curves.

The dimensional reduction of the 16-dimensional model may also have an interesting gravitational interpretation. We suggest that the degrees of freedom of the 8-dimensional theory can be related to the fields of twisted supergravity. The vacua of this theory are related to gravitational instantons with $Spin(7)$ holonomy. Particular solutions of this kind have recently been studied in [9][10].

Finally, one of our motivations is that a TQFT of a 3-form gauge field in higher dimensions is a quite attractive candidate for generating the M or F theory, as suggested by its central role in $D = 11$, $N = 1$ supergravity.

¹More precisely, we should say $SO(7)^\pm$, depending on the way $SO(7)$ is embedded in $SO(8)$.

2 The theory in 16 dimensions

Given a real 4-form in 16 dimensions, [1] indicates the existence of an interesting self-duality equation,

$$*G_4 = T_8 \wedge G_4, \quad (1)$$

that is, $G_{\mu\nu\rho\sigma} = T_{\mu\nu\rho\sigma\alpha\beta\gamma\delta} G_{\alpha\beta\gamma\delta}$, where the fully antisymmetric $SO(16)$ self-dual tensor T_8 is a singlet under a maximal subalgebra $SO(8) \times SO(7) \subset SO(15) \subset SO(16)$. The 4-form G_4 can be decomposed into a direct sum of terms that are irreducible under $SO(8) \times SO(7)$. One of the factors corresponds to a representation of dimension 455. The point is that this number is precisely the number of components of a 3-form gauge field in 16 dimension, defined modulo gauge transformations, $B_3 \sim B_3 + d\Lambda$, since $\binom{15}{3} = 455$. This tells us that one can interpret, (i), G_4 as the curvature of a 3-form B_3 , and, (ii), Eq.(1) as a 16-dimensional self-duality equation, which is $SO(8) \times SO(7)$ -invariant and allows one to determine B_3 , modulo gauge transformations. Moreover, the decomposition of this representation under $SO(8) \times SO(7)$ is [1] :

$$455 = (1, 35) \oplus (8, 21) \oplus (28, 7) \oplus (56, 1), \quad (2)$$

This decomposition is suggestive enough to indicate to us the way the various fields arising from the dimensional reduction in 8 dimensions can be arranged in group representations.

In 16 dimensions, we can consider the following $SO(8) \times SO(7)$ -invariant topological term:

$$\int_{M_{16}} T_8 \wedge G_4 \wedge G_4 \quad (3)$$

A BRST invariant gauge-fixing can be obtained by adding the following Q -exact term, which gives a 16-dimensional action:

$$\int_{M_{16}} d^{16}x \left\{ Q, \chi^{\mu\nu\rho\sigma} \left[(G_{\mu\nu\rho\sigma} - T_{\mu\nu\rho\sigma\alpha\beta\gamma\delta} G_{\alpha\beta\gamma\delta}) + \frac{1}{2} H_{\mu\nu\rho\sigma} \right] \right\} \quad (4)$$

Here Q is a standard topological BRST operator for the 3-form B_3 and the antighost $\chi^{\mu\nu\rho\sigma}$ is self-dual in the sense of Eq. (1). We do not write the complete action, which would necessitate the BRST invariant gauge-fixing of

the topological ghosts that occur in the definition of the topological BRST symmetry, as detailed for instance in [7].

Using the standard construction, Eq. (4) is expected to determine an action of the following form:

$$\int_{M_{16}} d^{16}x (G_{\mu\nu\rho\sigma} G^{\mu\nu\rho\sigma} - T_{\mu\nu\rho\sigma\alpha\beta\gamma\delta} G^{\mu\nu\rho\sigma} G^{\alpha\beta\gamma\delta} + \text{supersymmetric terms}) \quad (5)$$

In this action, $G_4 = dB_3$. The supersymmetric, i.e, ghost dependent terms, depend on the tensor T . Thus the $SO(16)$ covariant action is only $SO(8) \times SO(7)$ invariant, as the topological term in Eq. (3) depends on the given expression for T_8 .

We actually do not intend to study a 16 dimensional theory. Rather, we will shortly give attention to its possible descendants in 8 dimensions, which we will obtain from dimensional reduction arguments, and rearrangements of degrees of freedom in relevant group representations. The triality that exists in 8 dimensions will be useful.

The projection in 8 dimensions is suggested by the invariance of T_8 . At first sight, Eq. (2) suggests that the self-duality equation can be decomposed after reduction in 8 dimensions into self-duality equations for the curvatures of one 3-form, eight 2-forms, twenty-eight 1-forms, and fifty-six 0-forms. The above mentioned rearrangement means for instance that the latter fifty-six equations will be assembled into 7 Dirac like matrix equations, which mix the curvatures of these 56 scalar fields. Thus, we will use the possibility of identifying the 56 scalars as 7 spinors of $SO(8)$.

The counting is such that the number of all the projected self-duality equations is exactly the number of gauge invariant degrees of freedom of the forms. Indeed, the number of gauge invariant self-duality equation for the curvatures of 3-forms, 2-forms, 1-forms and 0-forms in eight dimensions are 35, 21, 7 and 1 respectively. Eventually, such self-duality equations can be enforced through a BRST invariant TQFT, where all propagators are fixed, and the gauge symmetries of forms are encoded in an equivariant way. As for Lorentz invariance in eight dimensions, [3] suggests that it will be at least reduced to $Spin(7) \subset SO(8)$, prior to an untwisting that could be allowed by triality. Let us recall that for the genuine Yang–Mills 8-dimensional TQFT, the full $SO(8)$ invariance is recovered after untwisting [3].

In the next section, we will detail these points and write self-duality equations that seem relevant to us in 8 dimensions.

3 Interacting gauged TQFTs in 8 dimensions

In the most naive approach, the fields that occur after the dimensional reduction of the abelian 3-form from 16 to 8 dimensions are classified in anti-symmetric representations of an internal global $SO(8)$ symmetry.

$$B_{\mu\nu\rho} \text{ (in } D = 16) \rightarrow (B_{\mu\nu\rho}, B_{\mu\nu}^a, A_\mu^{[ab]}, \Phi^{[abc]}) \text{ (in } D = 8) \quad (6)$$

The representations in which the 8-dimensional fields in the right hand side of Eq. (6) take their values, are of dimensions 1, 8, 28 and 56 respectively. The upper latin indices a, b, c, \dots denote the internal $SO(8)$ indices.

We are free to change the interpretation of these indices, as one often does in a topological field theory, a possibility that we understand as the essence of a twist. Moreover, we can gauge the internal $SO(8)$ symmetry by suitable redefinitions of the relation between the forms and the curvature. These redefinitions are constrained by the necessity of Bianchi identities for the curvatures.

In 8 dimensions, we can identify vector indices as spinor indices. We can interpret the eight 2-forms $B_{\mu\nu}^a$ as the components of a commuting spinorial 2-form field $B_{\mu\nu}^\alpha$, and the fifty-six 0-forms $\Phi^{[abc]}$ as the components of seven commuting 1/2-spin field $\Phi^{\alpha(i)}$, $1 \leq i \leq 7$, that is,

$$B_{\mu\nu}^a \rightarrow B_{\mu\nu}^\alpha \quad \Phi^{[abc]} \rightarrow \Phi^{\alpha(i)} \quad (7)$$

At this stage, $A_\mu^{[ab]}$ is a $SO(8)$ -valued gauge field. (We will shortly discuss a possible modification of this interpretation.) The spinorial index α in Eq.(7) runs from 1 to 8. The internal index i runs from 1 to 7 and can be interpreted as the index of a fundamental representation of dimension 7 of a given group, for instance $SO(7)$ or G_2 .

In view of the possible gauging of the internal covariance of $\Phi^{\alpha(i)}$ denoted by the index i , one can chose a preferred direction in $SO(8)$ and enforce the associated $SO(7)$ gauge symmetry. $A_\mu^{[ab]}$ can be further decomposed in a an $SO(7)$ -valued gauge field $A_\mu^{[ij]}$ with 21 components and a vector field $T_\mu^{[i]}$, which is valued in the fundamental representation of $SO(7)$. Analogously, we can split the eight 2-forms in $B_{\mu\nu}^\alpha$ as $B_{\mu\nu}^\alpha \sim (B_{\mu\nu}^i, B_{\mu\nu})$.

We thus have two possibilities,

$$B_{\mu\nu\rho} \text{ (in } D = 16) \rightarrow (B_{\mu\nu\rho}, B_{\mu\nu}^\alpha, A_\mu^{[ab]}, \Phi^{i\alpha}) \text{ (in } D = 8) \quad (8)$$

$$B_{\mu\nu\rho} \text{ (in } D = 16) \rightarrow (B_{\mu\nu\rho}, B_{\mu\nu}^i, B_{\mu\nu}, A_\mu^{[ij]}, T_\mu^i, \Phi^{i\alpha}) \text{ (in } D = 8) \quad (9)$$

Both choices allows us to write covariant self-dual equations for the fields and to build consistent eight-dimensional TQFTs. It must be noted that $SO(7)$ is a natural gauge group for eight-dimensional instantons. In fact, the eight-dimensional generalization of the 't Hooft symbols mix the left-over Lorentz symmetry $SO(7)$ with an internal $SO(7)$ symmetry group [6]. A third possibility exists, which is to reduce the gauge symmetry down to G_2 . In this case, 7 gauge fields among the $A^{[ij]}$ must be reinterpreted as 56 bosonic degrees of freedom, which merely amounts to a duplication of $\Phi^{i\alpha}$, a scheme that we will not discuss here.

Using the covariant derivative $D = d + A$ with respect to the $SO(8)$ or $SO(7)$ gauge field A , we now introduce covariant interactions by defining appropriately the relations between forms and curvatures. We take the following definitions for the fields in Eq.(8):

$$\begin{aligned} G_{\mu\nu\rho\sigma} &= \partial_{[\mu} B_{\nu\rho\sigma]} \\ G_{\mu\nu\rho}^\alpha &= D_{[\mu} B_{\nu\rho]}^\alpha \\ F_{\mu\nu}^{[ab]} &= \partial_{[\mu} A_{\nu]}^{[ab]} + A_{[\mu}^{[ac]} A_{\nu]c}^b \\ S_\mu^{\alpha(i)} &= \partial_\mu \Phi^{\alpha(i)} + A_\mu^{[ij]} \Phi^{\alpha(j)}, \end{aligned} \quad (10)$$

For the fields in Eq.(9), there is some flexibility, and we have the possibility of curvatures with more interactions ($C(A) = Tr_{SO(7)}(AdA + \frac{2}{3}AAA)$ is the Chern–Simons form):

$$\begin{aligned} G_{\mu\nu\rho\sigma} &= \partial_{[\mu} B_{\nu\rho]} + c_{ijk} F_{[\mu\nu}^{[ij]} K_{\sigma]}^k \\ G_{\mu\nu\rho}^\alpha &= (G_{\mu\nu\rho}^i = D_{[\mu} B_{\nu\rho]}^i + F_{[\mu\nu}^{[ij]} T_{\rho]j}^i, \quad G_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} + C(A)_{\mu\nu\rho}) \\ F_{\mu\nu}^{[ij]} &= \partial_{[\mu} A_{\nu]}^{[ij]} + A_{[\mu}^{[ik]} A_{\nu]k}^j \\ K_{\mu\nu}^i &= \partial_{[\mu} T_{\nu]}^i + A_{[\mu}^{[ij]} T_{\nu]j}^i + B_{\mu\nu}^i \\ S_\mu^{\alpha(i)} &= \partial_\mu \Phi^{\alpha(i)} + A_\mu^{[ij]} \Phi^{\alpha(j)} \end{aligned} \quad (11)$$

In both cases, the curvatures have been constructed from the requirement of fulfilling Bianchi identities, which are easy to check. The TQFTs that involve these curvatures must be defined in an 8-dimensional space with holonomy group smaller or equal to $Spin(7)$, in order to enable a self-duality equation for the Yang–Mills curvature.

The 8-dimensional self-duality equations that we choose for the fields in Eq.(10) are :

$$\begin{aligned}
G_{\mu\nu\rho\sigma} - \epsilon_{\mu\nu\rho\sigma\mu'\nu'\rho'\sigma'} G^{\mu'\nu'\rho'\sigma'} &= 0 \\
\epsilon_{\mu\nu\rho\sigma\tau abc} (\gamma^a \gamma^b \gamma^c)_\alpha^\beta G_{\rho\sigma\tau}^\alpha &= 0 \\
F_{\mu\nu}^{[ab]} - \Omega_{\mu\nu\rho\sigma} F^{\mu\nu[ab]} &= 0 \\
\gamma^\mu S_\mu^{\alpha(i)} &= 0
\end{aligned} \tag{12}$$

For the case of the field decomposition in Eq.(11), we have a more refined possibility, where f_{jk}^i stand for the structure coefficients of $SO(7)$:

$$\begin{aligned}
G_{\mu\nu\rho\sigma} - \epsilon_{\mu\nu\rho\sigma\mu'\nu'\rho'\sigma'} G^{\mu'\nu'\rho'\sigma'} &= 0 \\
\epsilon_{\mu\nu\rho\sigma\tau abc} (\gamma^a \gamma^b \gamma^c)_\alpha^\beta G_{\rho\sigma\tau}^\alpha &= 0 \\
F_{\mu\nu}^{[ij]} - \Omega_{\mu\nu\rho\sigma} F^{\mu\nu[ij]} &= \Phi^{\alpha[i} (\gamma_{[\mu} \gamma_{\nu]})_{\alpha\beta} \Phi^{j]\alpha} \\
K_{\mu\nu}^{[i]} - \Omega_{\mu\nu\rho\sigma} K^{\mu\nu j} &= f_{jk}^i \Phi^{\alpha j} (\gamma_{[\mu} \gamma_{\nu]})_{\alpha\beta} \Phi^{k\alpha} \\
\gamma^\mu S_\mu^{\alpha(i)} &= 0,
\end{aligned} \tag{13}$$

The equations for the curvatures of the one-forms are the octonionic equations used in [3], where $\Omega_{\mu\nu\rho\sigma}$ is the self-dual $Spin(7) \subset SO(8)$ -invariant tensor. Ω is defined from the octonionic structure coefficients c_{ijk} , with $\Omega_{8ijk} = c_{ijk}$. It allows one to irreducibly decompose in a $Spin(7)$ -invariant way the representation 28 of $SO(8)$ as the sum of the representations 21 and 7 of $SO(7)$. This explains why, as needed in 8 dimensions, the self-duality equations of the Yang–Mills curvature only count for seven independent equations.

The $(\gamma^\rho)_\alpha^\beta$ are the 8×8 eight-dimensional gamma matrices. Having arranged fields in spinorial representations is the key for having self-duality equations, which are first order equations. The existence of Ω follows from that of a covariantly constant spinor η , with $\Omega_{\mu\nu\rho\sigma} = \dagger \eta (\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) \eta$, which gives a reparametrization invariant definition of the closed 4-form Ω_4 . The spinor η exists when the space has a holonomy group $H \subset Spin(7)$.

The condition on G_4 is the obvious $SO(8)$ invariant self-duality condition for the abelian curvature of a 3-form in 8 dimensions. By enforcing this condition in a BRST invariant way, one gets a TQFT action, as explained in [3].

The second equation for the curvature G_3^α of the spinorial field B_2^α is $SO(8)$ -invariant, and deserves more explanation. It is analogous to a Rarita–Schwinger equation, but it involves a 2-form spinor, instead of the gravitino, which is one-form spinor. This equation counts as many conditions as there are degrees of freedom in B_2^α , modulo gauge transformations $B_2^\alpha \rightarrow B_2^\alpha + D\Lambda^\alpha$, since it is a two-form that linearly depends on G_3 . The rest of the degrees of freedom in B_2 must be gauge-fixed, using the techniques of equivariant gauge-fixing², in a way that generalizes the completion of the gauge fixing of A , once the seven gauge covariant conditions $F_{\mu\nu}^{[ab]} - \Omega_{\mu\nu\rho\sigma} F^{\rho\sigma[ab]} = 0$ have been imposed. Actually, the completion of the gauge-fixing of the 2-form B_2^α is inspired from that for a gravitino. It is:

$$(\gamma^\mu D_\mu)(\gamma^\nu B_{\nu\rho}) = 0 \quad (14)$$

It is then a simple exercise to show that the square $|\epsilon_{\mu\nu\rho\sigma\tau abc}(\gamma^a\gamma^b\gamma^c)G_{\rho\sigma\tau}|^2$ is essentially equal to $|G_{\mu\nu\rho\sigma}|^2 + |\gamma^\rho G_{\mu\nu\rho}|^2 + |\gamma^\nu\gamma^\rho G_{\mu\nu\rho}|^2$ plus a Feynman type gauge fixing for the 2-form gauge field B_2 , when Eq.(14) is enforced. The derivation is however lengthy and will be explained elsewhere. The important point is that one gets a $SO(8)$ -covariant propagator for the 2-form.

The other conditions on A and Φ give gauge interactions that are of interest, thanks to the couplings introduced in the definitions of the curvatures.

Using suitable Lagrange multipliers and antighosts, one can write a BRST-exact TQFT action, whose bosonic part is essentially the sum of the squares of these four conditions, that is,

$$\begin{aligned} \int d^8x (& |G_{\mu\nu\rho\sigma}|^2 + G_{\mu\nu\rho}^\alpha G_{\mu\nu\sigma}^\alpha + G_{\mu\nu\rho}^\alpha (\gamma^\rho\gamma^\sigma)_\gamma^\alpha G_{\mu\nu\sigma}^\gamma \\ & + G_{\mu\nu\rho}^\alpha (\gamma^\nu\gamma^\rho\gamma^\tau\gamma^\sigma)_{\alpha\beta} G_{\mu\tau\sigma}^\beta + |F_{\mu\nu}^{[ab]}|^2 + |S_\mu^{\alpha(i)}|^2 \\ & + \text{boundary and supersymmetric terms} \\ & + \text{ordinary gauge fixing terms}) \end{aligned} \quad (15)$$

Here we used the basic properties of the octonionic 4-form Ω [3], $|F_{\mu\nu}^{[ij]} + \Omega_{\mu\nu\rho\sigma} F^{\rho\sigma[ij]}|^2 = 3|F_{\mu\nu}^{[ij]}|^2 + \text{boundary terms}$, as well as gamma matrix identities.

²In the case of the system of Eq.(10), one must use the Batalin–Vilkoviski formalism, due to the non closure of the gauge transformation for a charged 2-form, and antifields are needed. In the case of the system of Eq.(11), owing to the presence of the field $T\mu^i$, the standard BRST technology is sufficient for completing the gauge-fixing.

The supersymmetric terms involve higher ghost interactions that are clumsy, but straightforward to derive.

We are in the presence of a very specific theory, which involves a charged 2-form, in a new and interesting way, with gauge interactions. By construction, it possesses a Q -symmetry. It is possible that, by using triality, this theory could be untwisted into theory which is invariant under the Poincaré supersymmetry, or, perhaps, only under part of it. This question will not be studied here.

We wish to emphasize that, when one uses the field decomposition of Eq.(11), the $SO(7)$ gauge symmetry of the non-abelian two-form follows from our definition of curvatures $G_3 = DB + FT$, $K = B_2 + DT$ with Bianchi identities $DG_3 = FK_2$, $DT_2 = G_3$, as in [8]. Eventually, this allows one to complete the gauge-fixing of ordinary gauge symmetries in the standard BRST way, without having to use the Batalin–Vilkoviski formalism, (one has in this case a first rank BV system), contrarily to the case of the decomposition of Eq.(10). Moreover, we see that the right-hand sides of the self-duality equations in Eq.(13) involve terms that are similar to those used in the four-dimensional TQFT for monopoles [5], which also derive from dimensional reduction of a theory in higher dimension (in this case an eight-dimensional one [3]). Finally, the presence of Chern–Simons terms in the curvatures of B_2 and B_3 give gauge symmetries that mix the gauge transformations of forms with Yang–Mills gauge transformation. If, in the process of dimensional reduction, one has the creation of an anomaly, these modifications of the curvatures and of the gauge transformations might be of interest for their cancellation.

3.1 A supergravity interpretation

We now turn to another suggestive interpretation of the fields A_μ^{ab} and Φ^{abc} . We can interpret the twenty-eight one-forms A_μ^{ab} as a spin connection for the 8-dimensional manifold, $\omega^{ab} = B^{ab}$, and the fifty-six zero-forms Φ^{abc} as the components of a constrained vielbein e_μ^a , which is appropriate for an 8-dimensional manifold with $Spin(7)$ holonomy, such that, $e_8^8 = 1$, $e_7^i = e_i^7 = \phi^i$ are described by 7 linear combinations of the 56 fields B^{abc} 's and e_j^i , $1 \leq i, j \leq 7$ are described by 49 other independent combinations of the

B^{abc} 's, with:

$$e_\mu^a = \begin{pmatrix} 1 & \phi^i \\ t\phi^i & e_j^i \end{pmatrix} \quad (16)$$

The decomposition of the 16-dimensional 3-form after reduction to eight dimensions is now of a purely gravitational nature :

$$(B_{\mu\nu\rho}, B_{\mu\nu}^\alpha, \omega_\mu^{ab}, e_\mu^a) \quad (17)$$

The interpretations of the topological ghosts and antighosts for e_μ^a are as the twisted gravitino of $N = 1$ supergravity in eight dimensions, adapted to the vielbein in Eq.(16).

For gauge-fixing the topological freedom for ω_μ^{ab} , we can choose the torsion free condition $T_{\mu\nu}^a = 0$, which allows one to eliminate ω as a function of e , in the standard way.

The manifold has $Spin(7)$ holonomy, in such a way that it contains the invariant closed 4-form Ω_4 . In this case, the gravitational instanton equation is just

$$\omega_\mu^{ab} = \Omega^{abcd}\omega_\mu^{cd}, \text{ that is, } \omega_\mu^{ab-} = 0, \quad (18)$$

which counts as $7 \times 8 = 56$ independent equations. It is relevant to use these 56 independent equations as the topological gauge functions for exhausting the topological gauge freedom in the vielbein e_μ^a in Eq.(16). The construction of topological gravity in 8 dimensions has been recently presented in [15].

Since we predict an action with a propagating metric, the 3-form gauge field $B_{\mu\nu\rho}$ and the spinorial fields $B_{\mu\nu}^\alpha$ are now subject to gravitational interactions by mean of a BRST exact action as in Eq.(15). These TQFT's are likely to describe invariants, which are related to the existence of gravitational instantons [9].

4 Dimensional reduction into renormalizable theories in 4 dimensions and below

We now consider further dimensional reductions, down to 4 dimensions. One motivation is of obtaining new renormalizable models, that contain abelian

monopoles, with coupling to supersymmetric matter. These models are in the spirit of the work of Seiberg and Witten, for getting effective theories that allow for unambiguous computations of microscopic theories that are purely non abelian. They include the models that Seiberg introduced.

As explained in [3], by dimensionally reducing in four dimensions the 8-dimensional octonionic Yang–Mills equation $*F = \Omega \wedge F$, one obtains the coupled non-abelian equations :

$$F_{\mu\nu}^+ = [\bar{M}, \Gamma_{\mu\nu} M], \quad \Gamma^\mu D_\mu M = 0 \quad (19)$$

Here, M^α is a Weyl spinor whose two complex commuting components are made from the gauge field components A_5, A_6, A_7, A_8 :

$$M^1 = A_5 + iA_6, \quad M^2 = A_7 + iA_8. \quad (20)$$

Thus $M = M^A T^A$ is valued in the same Lie algebra as $A = A^A T^A$. A further gauge fixing of the fields in the Cartan algebra is allowed by the topological gauge invariance, and one recovers from Eq. (19) the abelian Seiberg–Witten equation

$$F_{\mu\nu}^+ = \bar{M} \Gamma_{\mu\nu} M, \quad \Gamma^\mu D_\mu M = 0, \quad (21)$$

where M has the interpretation of a monopole. M and its topological ghosts build a chiral matter multiplet after untwisting. Let us explain the mechanism when the gauge symmetry is $SU(2)$. In this case, the projection in the Cartan algebra means $A_\mu^{(1)} = A_\mu^{(2)} = 0$ and $A_5^{(3)} = A_6^{(3)} = A_7^{(3)} = A_8^{(3)} = 0$, where the upper indices are $SU(2)$ indices. The Seiberg-Witten monopole with charge plus or minus one with respect to the abelian gauge field is simply given by the linear combination $M^\pm = \frac{1}{\sqrt{2}}(M^{(1)} \pm M^{(2)})$.

For our theory in 8 dimensions, if we choose the case of a $SO(8)$ gauge group, the Cartan algebra is made of 4 independent $U(1)$ symmetries. These symmetries will act with certain charges on the 56 scalars, when one decomposes these representation $\underline{56}$ of $SO(8)$ on the four $U(1)$. It gives other charges when the 56 scalars are assembled into seven spinors. If we restrict our-self to the maximal projection on one of the $U(1)$ subalgebra of the $U(1)^4$ Cartan algebra of $SO(8)$, only a certain number of the 56 scalars will remain coupled to the remaining abelian gauge field. It is interesting to observe that, depending on the $U(1)$ that one chooses, one gets the two possible values of the charge that are related by electromagnetic duality.

5 To matrix models

The strategy of obtaining exact solutions of $\mathcal{N} = 2$ gauge theories from embedding in higher dimensions naturally emerges also in the context of M-theory. One of the notable applications of this theory is indeed the solution of four-dimensional $\mathcal{N} = 2$ models via the analysis of suitable M-fivebrane backgrounds [11]. On the other hand, an explicit formulation of the M-theory is conjectured to be given by a matrix-model [12]. We recall that in [13] was explored the possibility of formulating a covariant action for the matrix strings from dimensional reduction of eight-dimensional Topological Yang-Mills Theory. It is interesting to observe that the dimensional reduction of our topological model on a two-torus together with a suitable gauge-fixing which set to zero all the fields but four components of the gauge field, say for example (A_4, A_5, A_6, A_7) , gives rise to the equations

$$F = \frac{i}{2}[\phi, \bar{\phi}], \quad \bar{\mathcal{D}}\phi = 0 \quad (22)$$

where $\mathcal{D} = D_6 + iD_7$, $F = \frac{i}{2}[\mathcal{D}, \bar{\mathcal{D}}]$, $\phi = A_4 + iA_5$ and the gauge connection is $A = A_6 + iA_7$. The same equations (22) can be derived from toroidal compactification of the Matrix theory in a suitable five-brane background, see Eq.(2.2) of [17], and describe the exact vacua of an $\mathcal{N} = 2$ four-dimensional model with an adjoint hypermultiplet. Moreover, it can be shown that the brane configuration of [11] corresponding to this model represents the same brane background as the Matrix theory setup [17].

Acknowledgments: We thank M. Bellon, B. Pioline and A. Tanzini for very useful discussions concerning this work.

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